Topology and the Dirac Spectrum for One-Flavor QCD

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Acknowledgments

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Relevant Papers

J.C. Osborn K. Splittorff and J. J. M. Verbaarschot, Chiral Symmetry Breaking and the Dirac Spectrum at Nonzero Chemical Potential, Phys. Rev. Lett. 94 (2005) 202001 [arXiv[hep-th/0501210]].

- J.J.M. Verbaarschot and T. Wettig, The Spectrum of the Dirac Operator for QCD with on flavor at fixed θ angle, Phys. Rev. D90 (2014) 070 [arXiv:1410.0883[hep-lat]].

P.H. Damgaard, Topology and the Dirac Operator Spectrum in Finite Volume Gauge Theory, Nucl. Phys. B556 327 (1999).

H. Leutwyler and A. Smilga, Spectrum of Dirac Operator and Role of Winding Number in QCD, Phys. Rev. D 46 (1992) 5607.

M. Creutz, One Flavor QCD, Ann. Phys. 322 (2007) 1518 [arXiv[hep-th/0609187]].

- J.J.M. Verbaarschot and T. Wettig, The Chiral Condensate of One-Flavor QCD and the Dirac Spectrum at $\theta = 0$, PoS LATTICE2014 (2014) 072 [arXiv:1412.5483 [hep-lat]].

Contents

I. Chiral Symmetry Breaking in One Flavor QCD

Chiral Condensate

Dirac Spectrum of One-Flavor QCD

One Flavor QCD

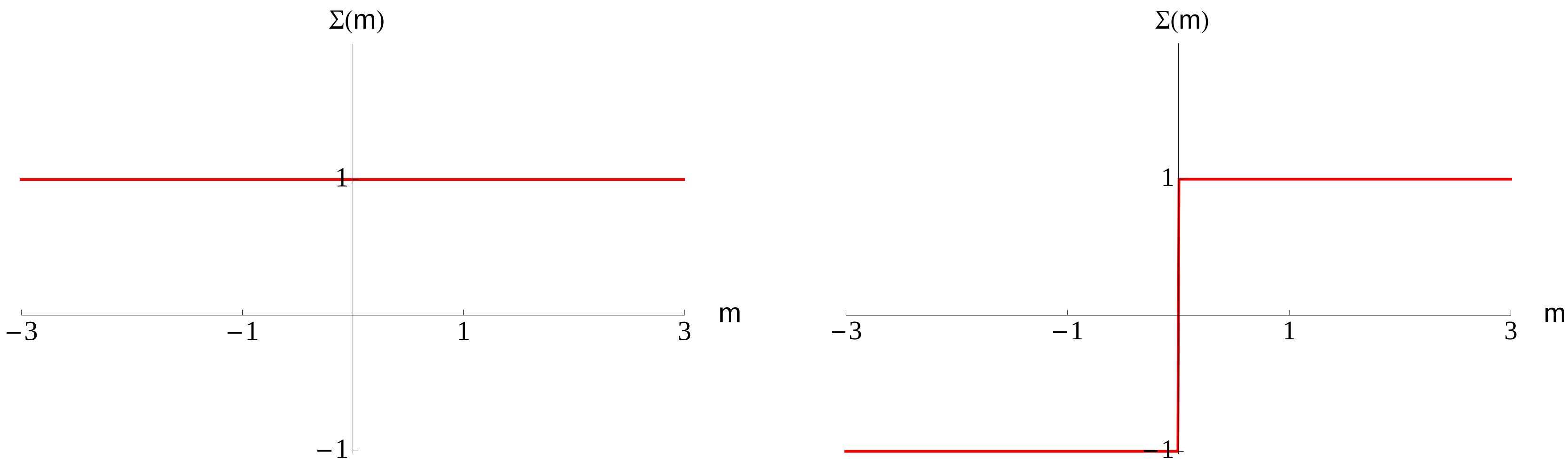
- Chiral symmetry is broken by the anomaly.
- There is no spontaneous symmetry breaking and there are no Goldstone bosons.
- The mass dependence of the one flavor QCD partition function is given by

$$Z = e^{mV\Sigma\cos\theta + O(m^2V)}.$$

- For $N_f = 2$ with spontaneous symmetry breaking, the mean field estimate of the partition function is given by (for $\theta = 0$)

$$Z = e^{|m|V\Sigma + O(m^2V)}.$$

Chiral Condensate



Behavior of the chiral condensate for $N_f = 1$ (left) and $N_f = 2$ (right).

$$\Sigma(m) = -\langle \bar{q} q \rangle = \frac{d}{dm} \log Z(m)$$

The goal of this talk is to explain this behavior in terms of the Dirac spectrum.

Dirac Spectrum

The spectral density of the Dirac operator is given by

$$\rho(x,m,\theta) = \frac{\sum_\nu e^{i\nu\theta} Z_\nu(m)\rho_\nu(x,m)}{\sum_\nu e^{i\nu\theta} Z_\nu(m)}.$$

The partition function at fixed topology is given by

$$Z_\nu(m) = m^\nu \langle \prod_k (\lambda_k^2 + m^2) \rangle_\nu.$$

For $m < 0$ the partition function is not positive definite and $\rho(x, m < 0, \theta)$ may become negative.

Unless stated otherwise we will take $\theta = 0$ in the remainder of the talk but vary the quark mass from positive to negative values.

Could it be that $\rho(0) = 0$?

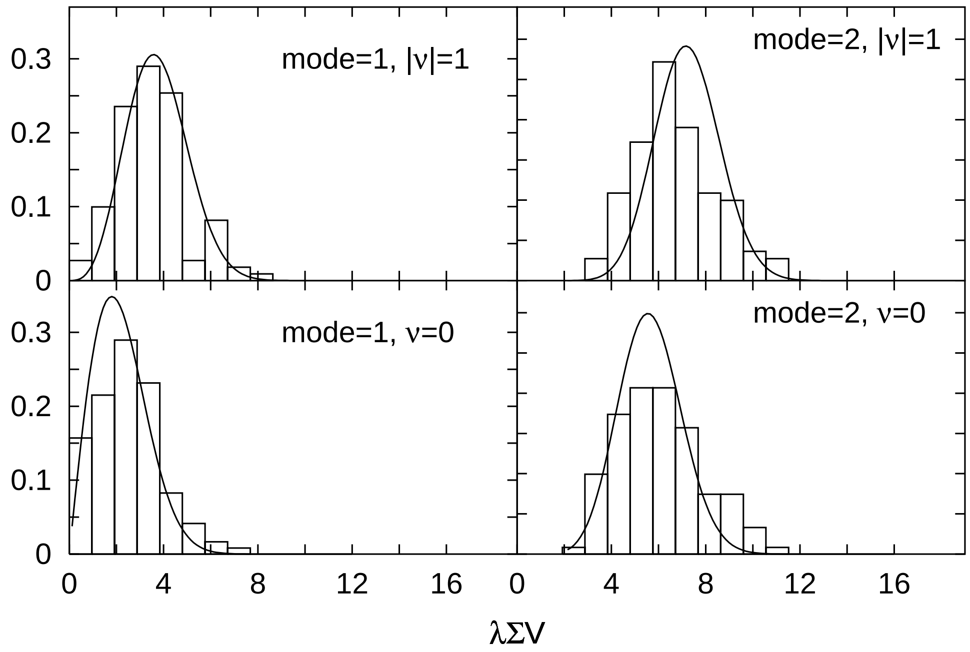
Banks-Casher relation

$$\rho(m = 0, \lambda = 0) = \frac{V}{\pi}(\langle \bar{q}q \rangle_{m=0_-} - \langle \bar{q}q \rangle_{m=0_+}).$$

We could conclude that because of the absence of a discontinuity in the chiral condensate we have $\rho(0) = 0$. Creutz-2006

We will see that the solution to this puzzle is much more subtle than this.

Lattice results for one-flavor QCD



Distribution of the lowest two Dirac eigenvalues for QCD with one flavor compared to the result from chiral random matrix theory (solid curve).

Degrand-Hoffmann-Schäfer-Liu-2006

What Can We Learn from Lattice Dirac Spectra?

- The smallest Dirac eigenvalues of one-flavor QCD behave in exactly the same was as the Dirac spectrum of QCD and QCD-like theories with spontaneously broken chiral symmetry.
- Although the Dirac spectrum at fixed topology has all signatures of spontaneous chiral symmetry breaking, it should emulate a chiral condensate that is due to explicit chiral symmetry breaking.

The Microscopic Domain of QCD

We will do our calculations in the microscopic domain of QCD, and the explicit results we quoted before were already in this domain.

In this domain, also know as the ϵ -domain, the quark mass and the Dirac eigenvalues scale as

$$m \sim \frac{1}{V}, \qquad \lambda \sim \frac{1}{V}.$$

Correction terms will enter when $m, \lambda \approx 1/\Lambda_{\rm QCD}\sqrt{V}$.

In this domain, the spectral density can be evaluated analytically.

Spectral Density at Fixed ν for $N_f = 1$

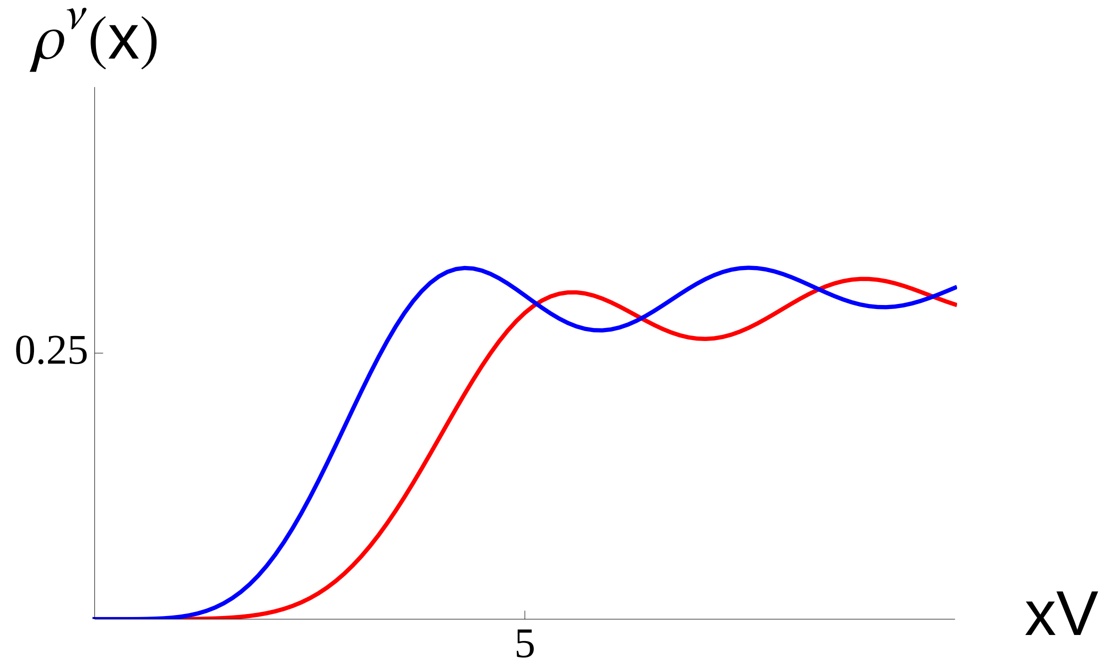
The one-flavor spectral density in the ϵ-domain is given by

$$\begin{aligned}\rho_\nu(\lambda, m) &= \frac{\hat{x}}{2}(J_\nu^2(\hat{x}) - J_{\nu+1}(\hat{x})J_{\nu-1}(\hat{x})) + |\nu|\delta(\hat{x}) \\ &\quad - \frac{\hat{x}}{\hat{m}^2 + \hat{x}^2}\left[\hat{x}J_\nu(\hat{x})J_{\nu+1}(\hat{x}) - \hat{m}\frac{I_{\nu+1}(\hat{m})}{I_\nu(\hat{m})}J_\nu^2(\hat{x})\right].\end{aligned}$$

Damgaard-Osborn-Toublan-JV-1999

$$\hat{x} \equiv \lambda\Sigma V, \qquad \hat{m} \equiv m\Sigma V$$

Microscopic Spectral Density at fixed ν



The one microscopic spectral density for $\nu = 2$ and $mV = 1$ (red) compared to the quenched result for $\nu = 2$ (blue).

II. Role of Topology

Decomposition in Topological Sectors

Is the Chiral Condensate Due to Zero Modes

Chiral Condensate at $\theta = 0$

Topological Decomposition

$$Z(m,\theta) = e^{mV\Sigma\cos\theta} = \sum_\nu e^{i\nu\theta} Z_\nu(m).$$

The partition function at fixed ν is given by (Leutwyler-Smilga-1991)

$$Z_\nu(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta e^{mV\Sigma\cos\theta} = I_\nu(mV\Sigma).$$

Asymptotic behavior of Bessel functions

$$I_\nu(\hat{m}) \sim \begin{cases} \frac{1}{\sqrt{2\pi\hat{m}}} e^{|\hat{m}|-\nu^2/2|\hat{m}|}, & \hat{m} > 0, \\ \frac{(-1)^\nu}{\sqrt{2\pi|\hat{m}|}} e^{|\hat{m}|-\nu^2/2|\hat{m}|}, & \hat{m} < 0 \end{cases}.$$

Notice that there are exponential cancellations to recover the partition function at $\theta = 0$ for $m < 0$.

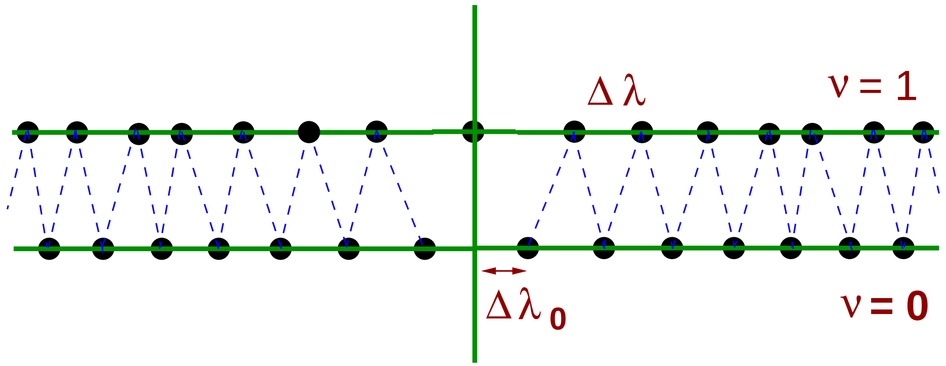
The topological susceptibility is equal to $\chi = mV\Sigma$.

Can the Chiral Condensate be Due to the Zero Modes?

If we take the chiral limit before the thermodynamic limit then

$$-\langle \bar{q}q \rangle = \frac{1}{VZ(m)} \left\langle \sum_\nu \sum_k \frac{1}{i\lambda_k + m} m^{|\nu|} \prod_k (i\lambda_k + m) \right\rangle$$

$$\underset{m\to 0}{=} \frac{1}{V} \frac{\left\langle \prod_{\lambda_k \neq 0} i\lambda_k \Big|_{\nu=1} \right\rangle}{\left\langle \prod i\lambda_k |_{\nu=0} \right\rangle} + \frac{1}{V} \frac{\left\langle \prod_{\lambda_k \neq 0} i\lambda_k \Big|_{\nu=-1} \right\rangle}{\left\langle \prod i\lambda_k |_{\nu=0} \right\rangle} \approx \frac{2}{V\Delta\lambda_0}$$



The nonzero eigenvalues shift on average by $\nu\Delta\lambda/2$.

Even in the chiral limit, the value of the chiral condensate is due to the nonzero modes.

Contribution of Zero Modes for $mV\Sigma \gg 1$

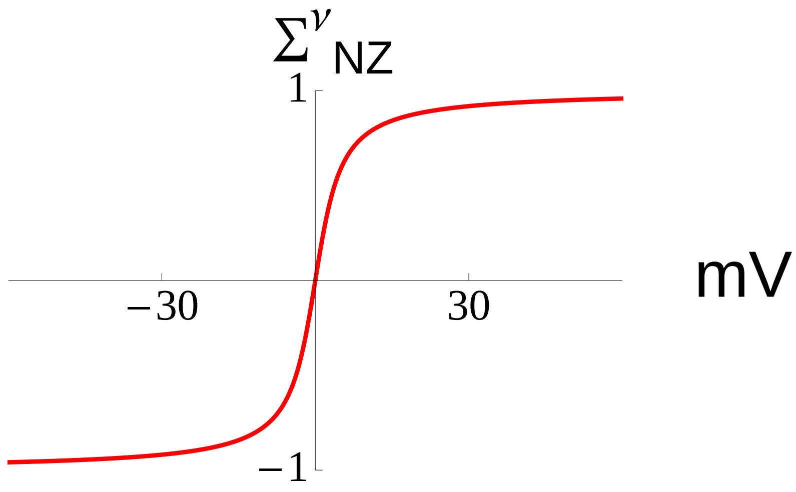
If the thermodynamic limit is taken before the chiral limit we have that $mV\Sigma \gg 1$

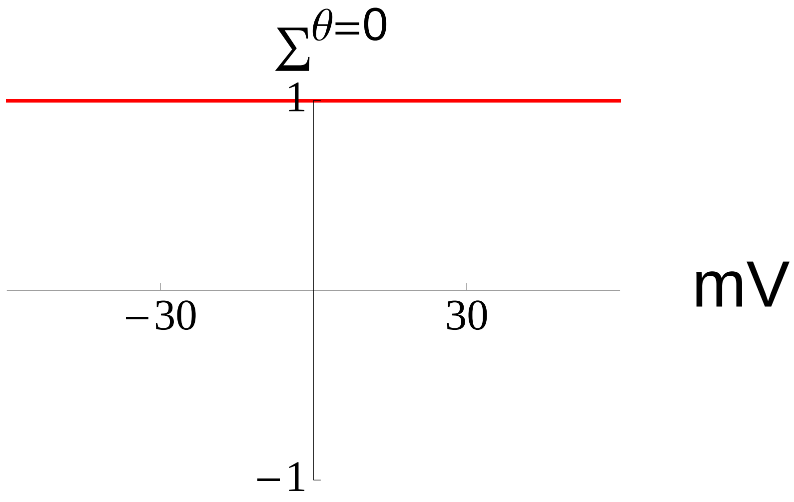
$$
\begin{aligned}
-\langle \bar{q}q \rangle &= \frac{1}{V}\frac{\sum_\nu \frac{|\nu|}{m} e^{-\nu^2/2|m|V\Sigma}}{\int d\nu e^{-\nu^2/2|m|V\Sigma}} \\
&= \frac{1}{Vm}\frac{\sum_\nu |\nu| e^{-\nu^2/2|m|V\Sigma}}{\int d\nu e^{-\nu^2/2|m|V\Sigma}} \\
&= \mathrm{sign}(m)\frac{2\Sigma}{\sqrt{\pi 2|m|V\Sigma}}.
\end{aligned}
$$

To get a constant chiral condensate we need the contribution of the nonzero modes.

This analysis is incorrect for $m < 0$ because of the exponential cancellations.

Chiral Condensate for $N_f = 1$





Mass dependence of the chiral condensate due to the nonzero modes for $\nu = 2$ (left) and the mass dependence of the chiral condensate at $\theta = 0$. Note that $\Sigma^{\nu}_{NZ}(m) = \Sigma^{\nu}(m) - \frac{|\nu|}{mV}$.

For $m < 0$, the negative values of $\Sigma^{\nu}(m)$ should average to a positive number. This is possible because the weight $Z_{\nu}(m)$ is not positive definite.

Silver Blaze Problem

- The chiral condensate of one flavor QCD stays the same when the quark mass crosses zero, but the spectral density of the Dirac operator is strongly affected.
- What is the solution of this “Silver Blaze Problem”? (© Cohen-2003)
- A similar problem arose in QCD at nonzero chemical potential where the chiral condensate remains constant while the Dirac spectrum is strongly altered by the chemical potential.
- It was shown that the chiral condensate can remain constant when the mass crosses a line of eigenvalues if the spectral density is oscillating with an amplitude that diverges exponentially with the volume and a period proportional to the inverse volume. This mechanism was discovered for a chiral random matrix theory at nonzero chemical potential. Osborn-Splittorff-JV-2005

Let us see how it can work for QCD with one flavor.

III. Role of Dynamical Quarks

Decomposition into Quenched and Dynamical Part

How To Obtain a Constant Chiral Condensate

Decomposition of the Spectral Density

- To confirm if the OSV mechanism holds we have to calculate the spectral density of the Dirac operator for $N_f = 1$ QCD.
- Actually this can be done analytically in the ϵ domain of QCD. The result can be expressed as a simple one dimensional integral.

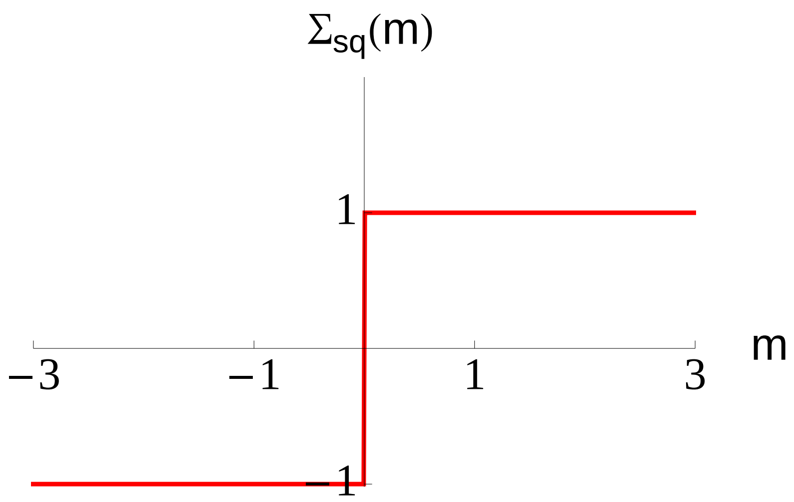
We decompose

$$\rho(\lambda, m) = \begin{cases} \rho_{\text{zm}}(\lambda, m) + \rho_{\text{nz}}(\lambda, m)\,, \\ \rho_{\text{sq}}(\lambda, m) + \rho_{\text{osc}}(\lambda, m) + \rho_{\,\delta\,\text{zm}}(\lambda, m)\,, \end{cases}$$

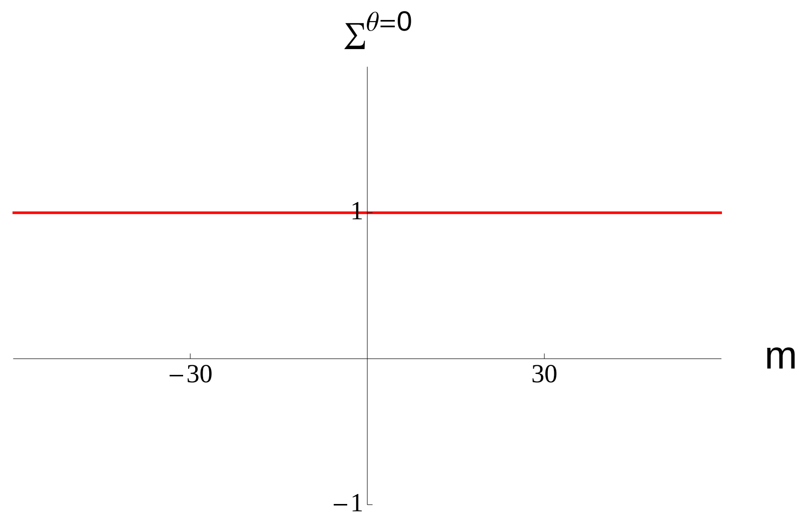
where

$$\begin{aligned} \rho_{\text{sq}}(\lambda, m) &= \rho(\lambda, |m|)\,, \\ \rho_{\text{osc}}(\lambda, m) &= \rho_{\text{nz}}(\lambda, m) - \rho_{\text{nz}}(\lambda, |m|)\,, \\ \rho_{\,\delta\,\text{zm}}(\lambda, m) &= \rho_{\text{zm}}(\lambda, m) - \rho_{\text{zm}}(\lambda, |m|)\,. \end{aligned}$$

How to get a Constant Chiral Condensate?



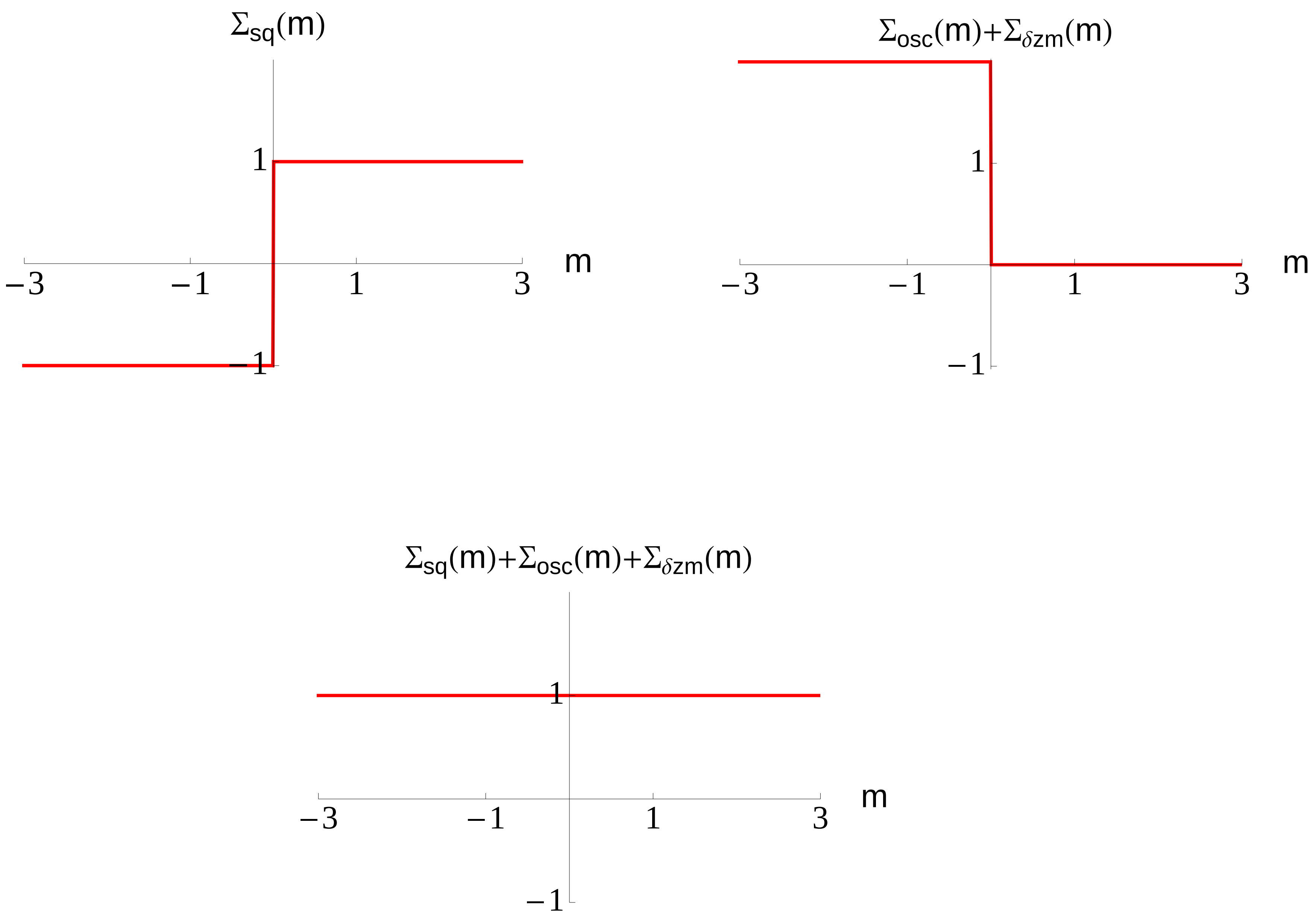
Behavior of the chiral condensate due to a line of eigenvalues for the quenched theory at $\theta = 0$.



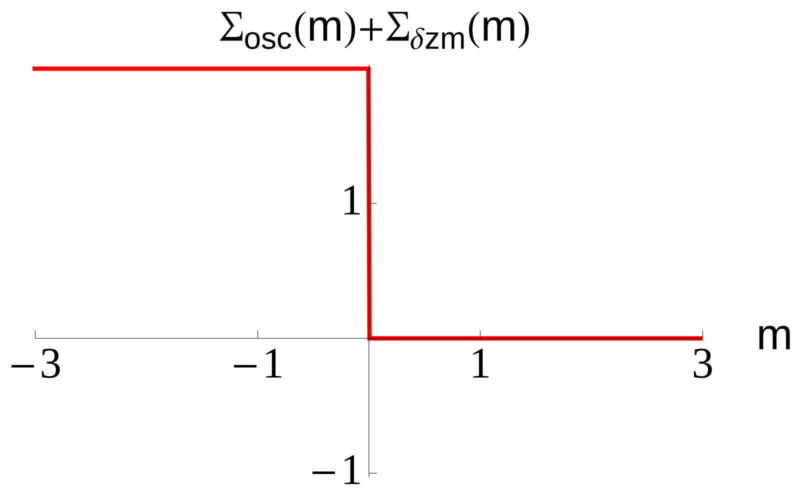
Behavior of the chiral condensate due to a line of eigenvalues for the one flavor theory at $\theta = 0$.

This implies that the not positive definite measure should give a correction to the spectral density that results in a mass dependence of the chiral condensate given by $\Sigma_{\rm osc}(m) + \Sigma_{\delta zm} = 2\theta(-m)$.

OSV Mechanism in Pictures



How can this be Generated by a Spectral Density?



$$2\theta(-m) = \int d\lambda \frac{\rho_{\rm osc}(\lambda, m) + \rho_{\delta{\rm zm}}(\lambda, m)}{i\lambda - m}.$$

What is $\rho_{\rm osc}(\lambda, m) + \rho_{\delta rmzm}(\lambda, m)$?

Hint,

$$\theta(m) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\tau \frac{e^{im\tau + m\epsilon}}{\tau - i\epsilon}.$$

Solution satisfying $\rho(\lambda) = \rho(-\lambda)$

$$\rho_{\rm osc}(\lambda, m) + \rho_{\delta {\rm zm}}(\lambda, m) = \frac{1}{\pi}(e^{iV\lambda - Vm} + e^{-iV\lambda - Vm})$$

$$\int_{-\infty}^{\infty} d\lambda \frac{1}{i\lambda - m}\frac{1}{\pi}(e^{V(i\lambda - m)} + e^{V(i\lambda - m)}) = 2\theta(-m) - 2\theta(m)e^{-2Vm}$$

satisfying $\rho(\lambda) = \rho(-\lambda)$ is given by Therefore, the chiral condensate due to the spectral density

$$\rho(\lambda, m) = \frac{1}{\pi}(1 - e^{V(i\lambda - m)} - e^{-V(i\lambda + m)}).$$

does not have a discontinuity across $m = 0$.

What is the Most General Class of solutions?

At least in the thermodynamical limit, the solution for the spectral density is not unique. Another solution that gives $2\theta(-m)$ in the thermodynamic limit is given by

$$\rho(x, m) = -\frac{4}{\pi}\frac{x^2}{x^2+m^2}\int_0^1 \frac{tdt}{\sqrt{1-t^2}} e^{-2mVt^2} J_1(2xVt)$$

However, we have the stronger requirement that also in the microscopic domain the contribution to the condensate is given by $2\theta(-m)$.

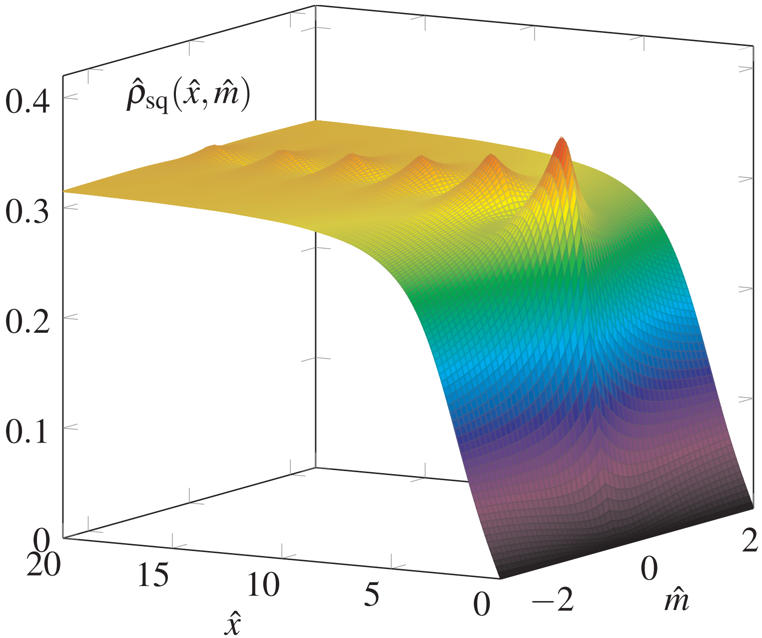
IV. One Flavor QCD

Spectral Density

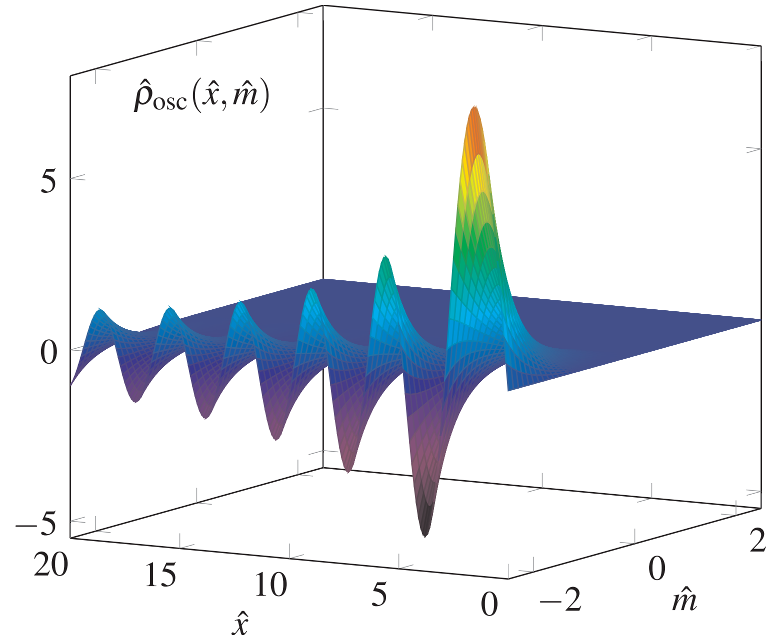
Zero Modes

Solution of Silver Blaze Problem

The Dirac Spectrum for $N_f = 1$ at $\theta = 0$



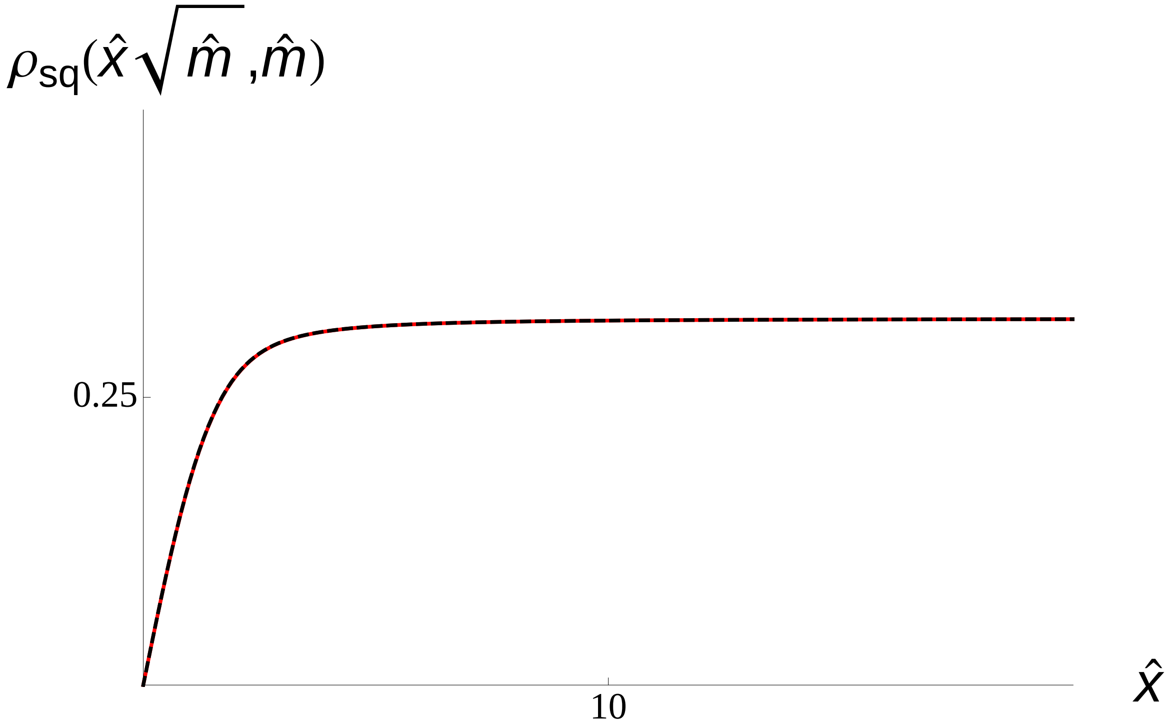
The sign quenched part of the spectral density at $\theta = 0$, $\hat{\rho}_{\rm sq}(\hat{x}, \hat{m})$.



The oscillating part of the spectral density at $\theta = 0$, $\hat{\rho}_{\rm osc}(\hat{x}, \hat{m})$.

$$\hat{\rho}_{\rm nz}(x, m) = \frac{1}{\pi}\int_0^1 \frac{e^{-2mVt^2}dt}{t\sqrt{1-t^2}} J_1(2xVt) - \frac{2}{\pi}\frac{x}{x^2+m^2}\int_0^1 \frac{e^{-2mVt^2}dt}{\sqrt{1-t^2}} \times\left[xtJ_1(2xVt) + m(1-2t^2)J_0(2xVt)\right] .$$

Asymptotic Scaling for $m > 0$

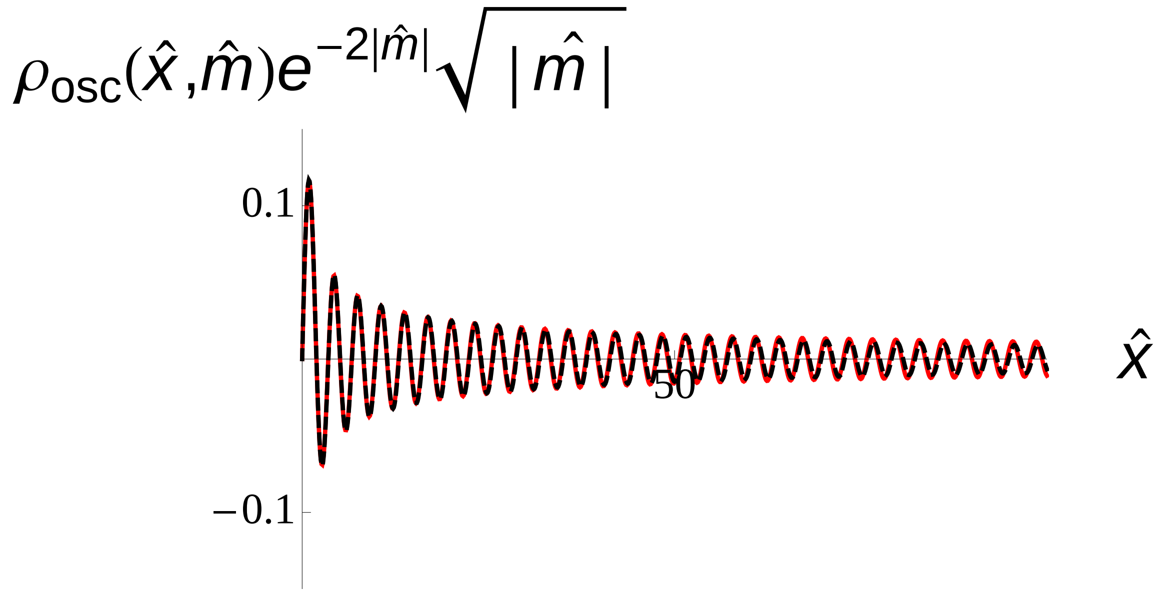


Spectral density for $mV\Sigma \gg 1$.

$$\rho_{\rm sq}(x\sqrt{m}, m > 0) \sim \frac{xV}{\sqrt{2\pi mV}} e^{-Vx^2/4m}[I_0(Vx^2/4m) + I_1(Vx^2/4m)]$$

Wettig-JV-2014

Asymptotic Scaling for $m < 0$



Spectral density for $mV\Sigma \ll -1$.

$$\rho_{\mathrm{osc}}(x, m < 0) \sim \frac{e^{2|m|V}}{\sqrt{8\pi|m|V}} J_1(2xV).$$

Wettig-JV-2014

Spectral Density due to Zero Modes

Contribution from zero modes — Damgaard-1999

$$\rho_{\mathrm{zm}}(x,m) \quad = \quad e^{-mV}\sum_{\nu}|\nu|I_\nu(mV)\delta(x) = e^{-mV}(I_0(mV)+I_1(mV))\delta(x).$$

Spectral density becomes exponentially large for $m < 0$.

Chiral Condensate

The chiral condensate can be obtained by integration over the spectral density

$$\Sigma(m) = \frac{1}{V} \int_{-\infty}^{\infty} \frac{2m\rho(\lambda, m)}{\lambda^2 + m^2}.$$

Role of the Zero Modes

For $m < 0$ the contribution from the zero modes diverges in the thermodynamic limit as

$$\Sigma_{\delta \mathrm{zm}}(m) \underset{\substack{V\to\infty \\ m<0}}{\sim} \frac{e^{2|mV|}}{\sqrt{8\pi|mV|^3}}.$$

This contribution cancels against a similar contribution from the nonzero modes. Kanazawa-Wettig-2012

Indeed we do find the asymptotic behavior JV-Wettig-2014

$$\Sigma_{\mathrm{osc}}(m) \underset{\substack{V\to\infty \\ m<0}}{\sim} -\frac{e^{2|mV|}}{\sqrt{8\pi|mV|^3}}.$$

Cancellation to All Orders

Actually, this cancellation takes place to all orders in $1/mV\Sigma$.

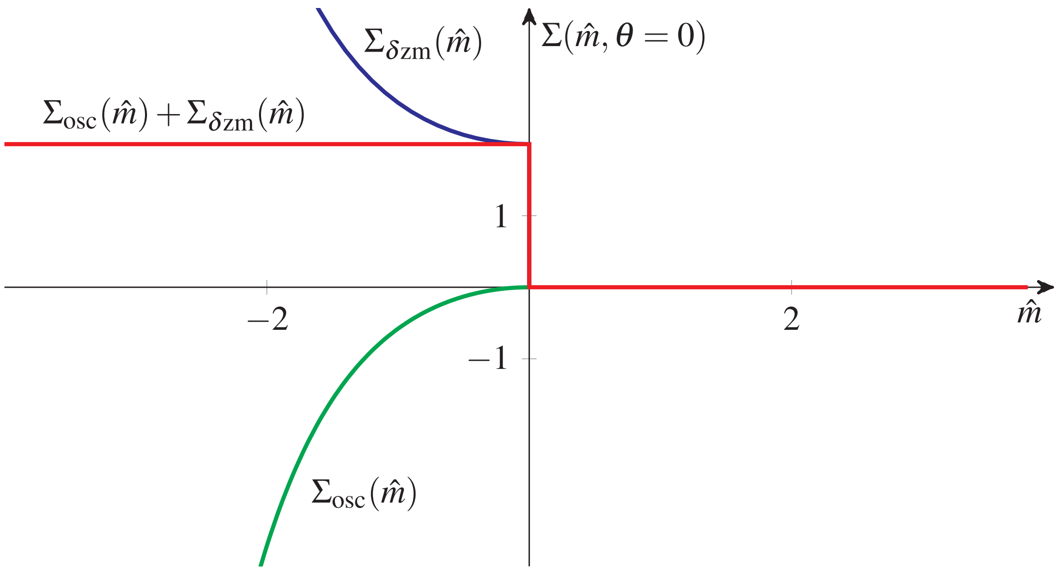
Using the exact analytical expressions, we have shown the identity

$$\Sigma_{\rm osc}(m) + \Sigma_{\delta{\rm zm}}(m) = 2\theta(-\hat{m}).$$

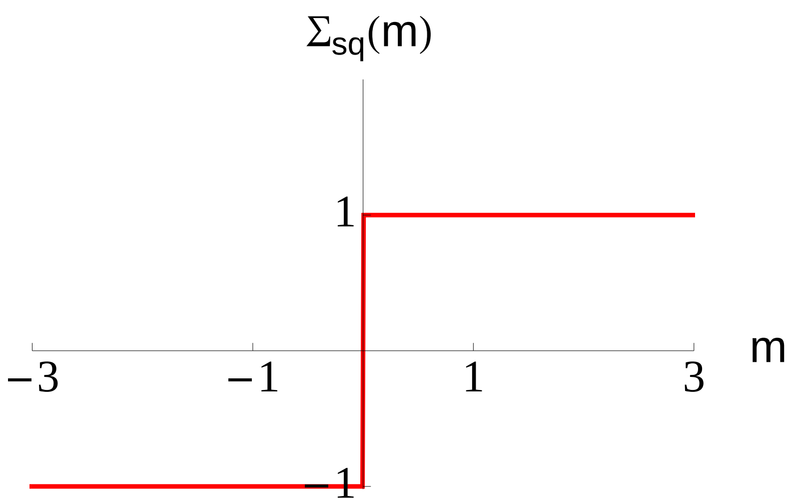
JV-Wettig-2014

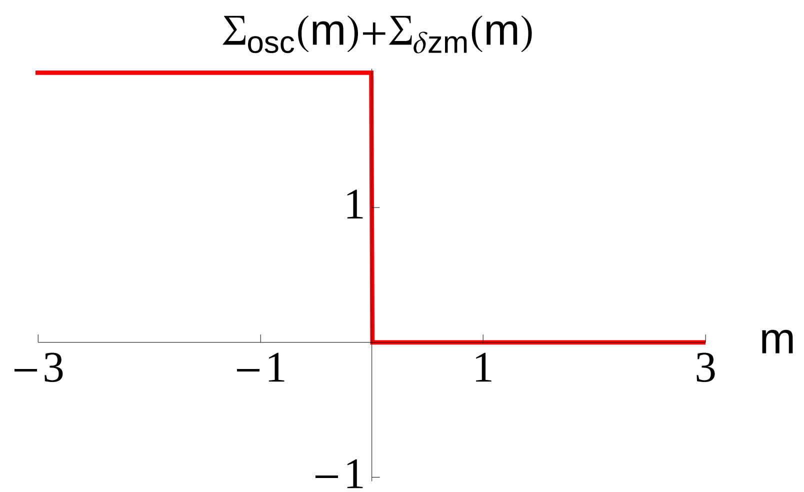
This is valid for $m\Lambda_{\rm QCD} \ll 1/\sqrt{V}$

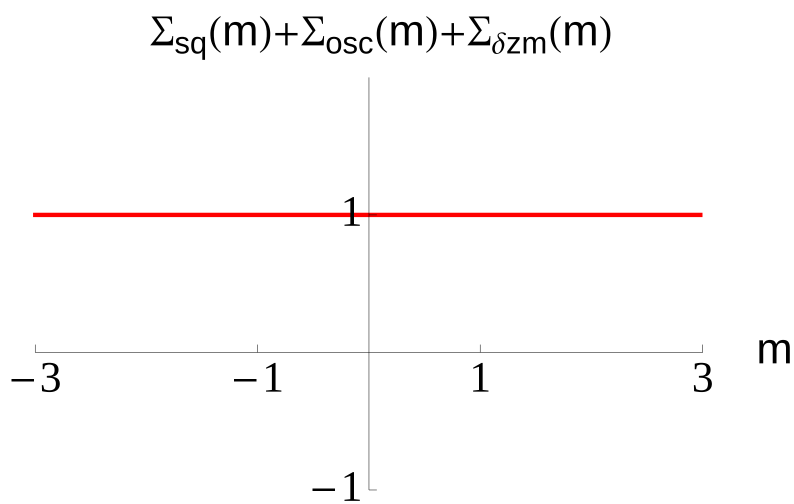
Total Contribution due to the Sign of the Mass



Solution of the Silver Blaze Problem







V. Conclusions

- The chiral condensate in the massless limit of one-flavor QCD is nonzero because of the zero modes but its value is determined by the nonzero modes.

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V. Conclusions

- The chiral condensate in the massless limit of one-flavor QCD is nonzero because of the zero modes but its value is determined by the nonzero modes.
- One flavor QCD has a Silver Blaze problem when the chiral condensate remains constant while the quark mass crosses a line of eigenvalues.
- From QCD at nonzero chemical potential we have learned that the solution of the Silver Blaze problem requires an oscillating spectral density with period $\sim 1/V$ and an amplitude that grows exponentially with the volume.

V. Conclusions

- In the ϵ domain of QCD we have obtained simple exact analytical expressions for the eigenvalue density of the Dirac operator at $\theta = 0$ and $\theta = \pi$. However, in this case the oscillating part of the spectral density gives an exponentially increasing contribution to the chiral condensate.

V. Conclusions

- In the ϵ domain of QCD we have obtained simple exact analytical expressions for the eigenvalue density of the Dirac operator at $\theta = 0$ and $\theta = \pi$. However, in this case the oscillating part of the spectral density gives an exponentially increasing contribution to the chiral condensate.
- The zero modes are essential for the continuity of the chiral condensate. Their exponentially increasing contribution is canceled exactly against the contribution from the nonzero modes.